Calculating the shadow of a black hole

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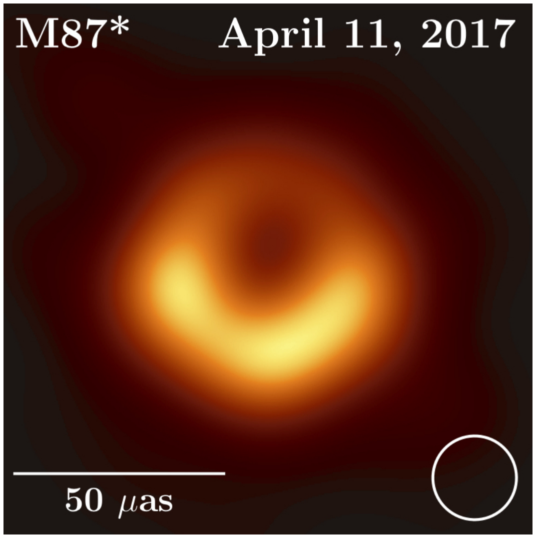


Event Horizon Telescope

Texas Tech University, USA, 3 November 2020

Outline

1. Lensing by a Schwarzschild black hole
   - photon sphere
   - multiple imaging
   - shadow
2. Lensing by a Kerr black hole
   - photon region
   - shadow
3. Observability of the shadow
   - the Event Horizon Telescope
   - Sgr A and Sgr A*
   - M 87 and M 87*
   - the picture of M 87*

VP: Gravitational lensing from a spacetime perspective,
Liv. Rev. Relativity 7, 9 (2004),
https://link.springer.com/article/10.12942/lrr-2004-9

Preliminary Remark

Observational evidence for the existence of black holes

- **stellar black holes**

  4 to 100 $M_\odot$

  X-ray binaries such as Cyg X-1

  LIGO events

- **supermassive black holes**

  10^6 to 10^{10} $M_\odot$

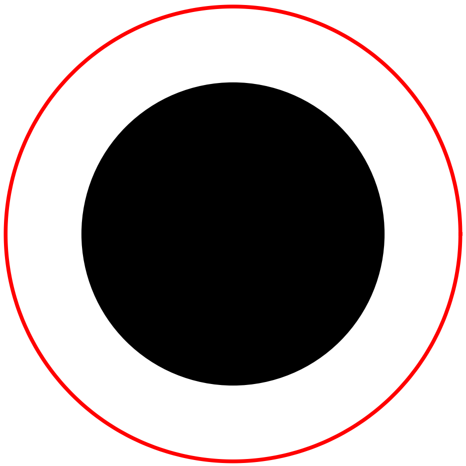
  exist at the centres of galaxies, such as those associated with the radio sources Sgr A* and M87*

1. Lensing by a Schwarzschild black hole

Schwarzschild(-Droste) metric:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2m}{r}\right)c^2dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

where $m = \dfrac{GM}{c^2}$



Horizon:

$r_S = 2m =$ **Schwarzschild radius**

Light sphere (photon sphere)

$$\frac{3}{2}r_S = 3m$$

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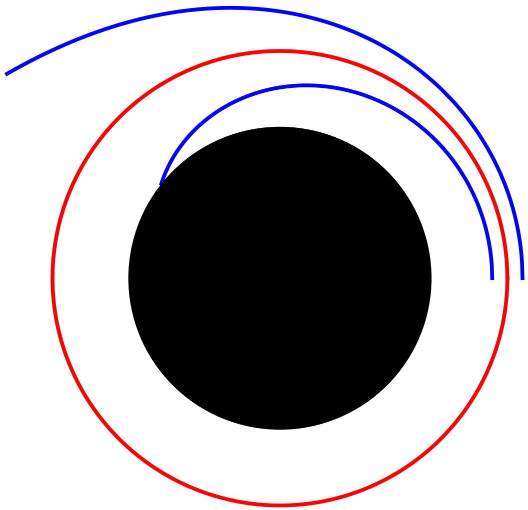

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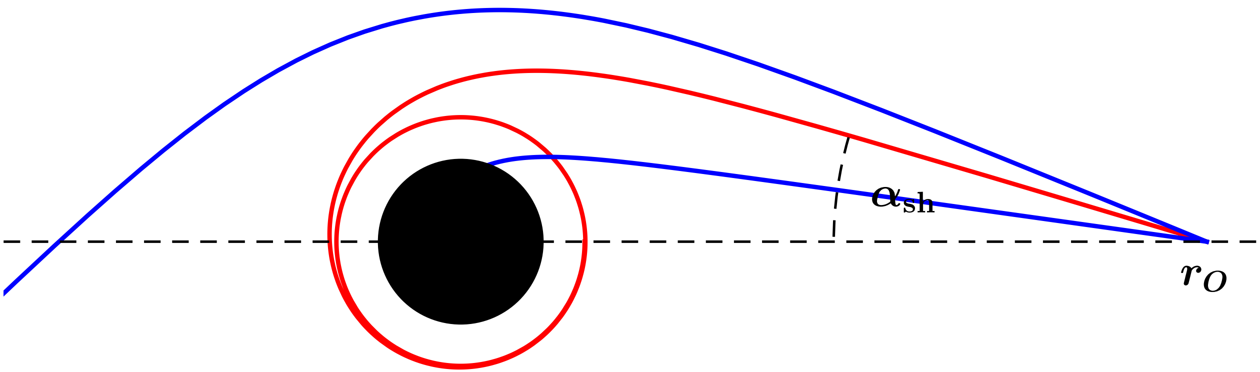
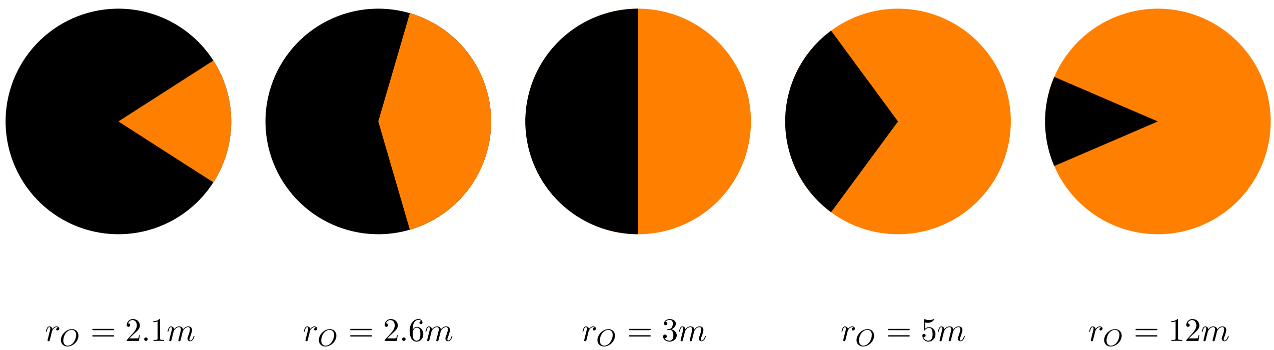


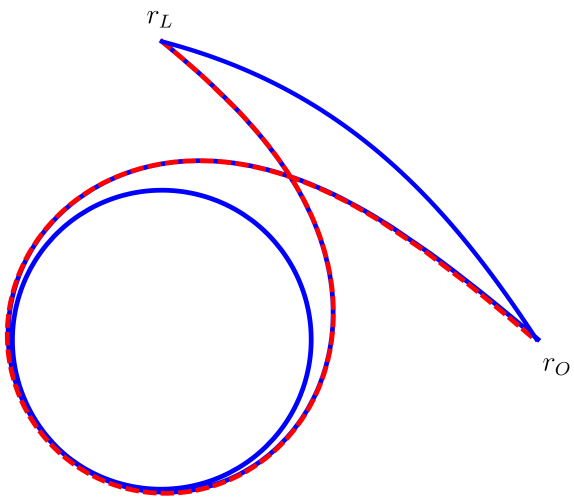
Angular radius α_{sh} of the shadow of a Schwarzschild black hole:

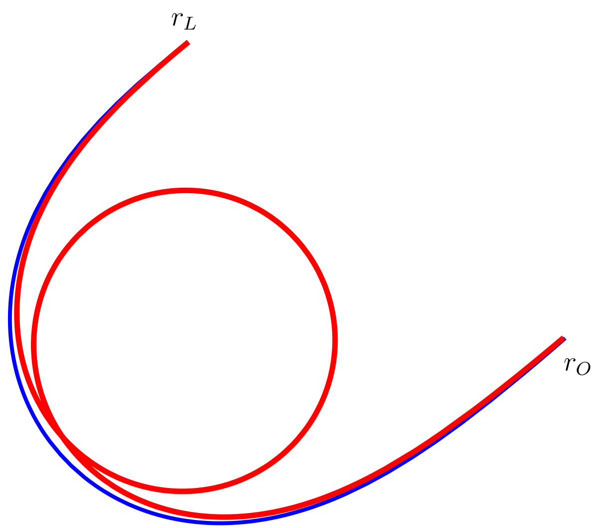
$$\sin^2\alpha_{\mathrm{sh}} = \frac{27\, r_S^2\,(r_O - r_S)}{4\, r_O^3} = \frac{27\, m^2}{r_O^2}\Big(1 - \frac{2m}{r_O}\Big)$$

J. L. Synge, Mon. Not. R. Astr. Soc. 131, 463 (1966)

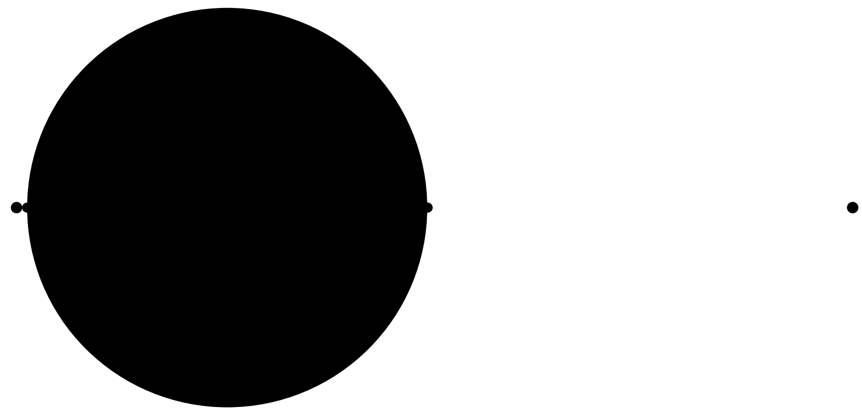


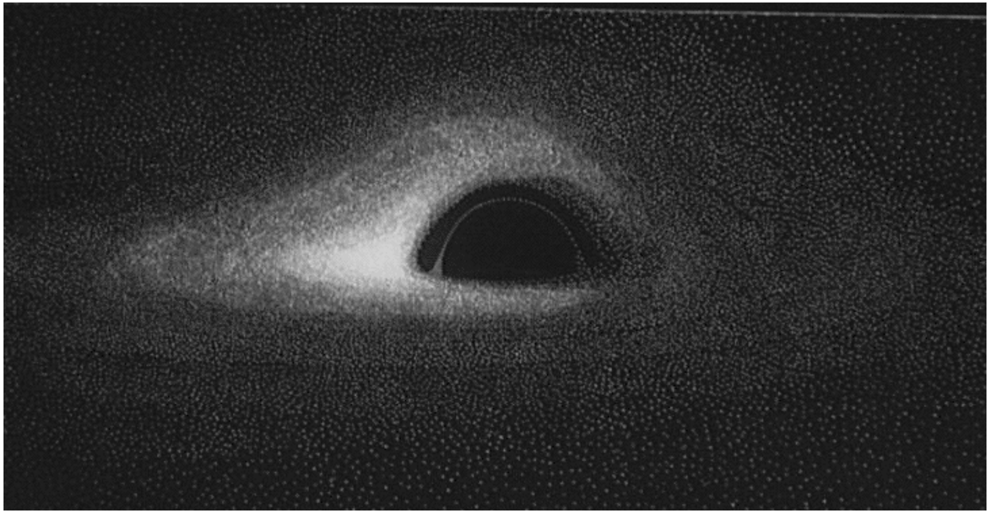
Schwarzschild black hole produces infinitely many images:



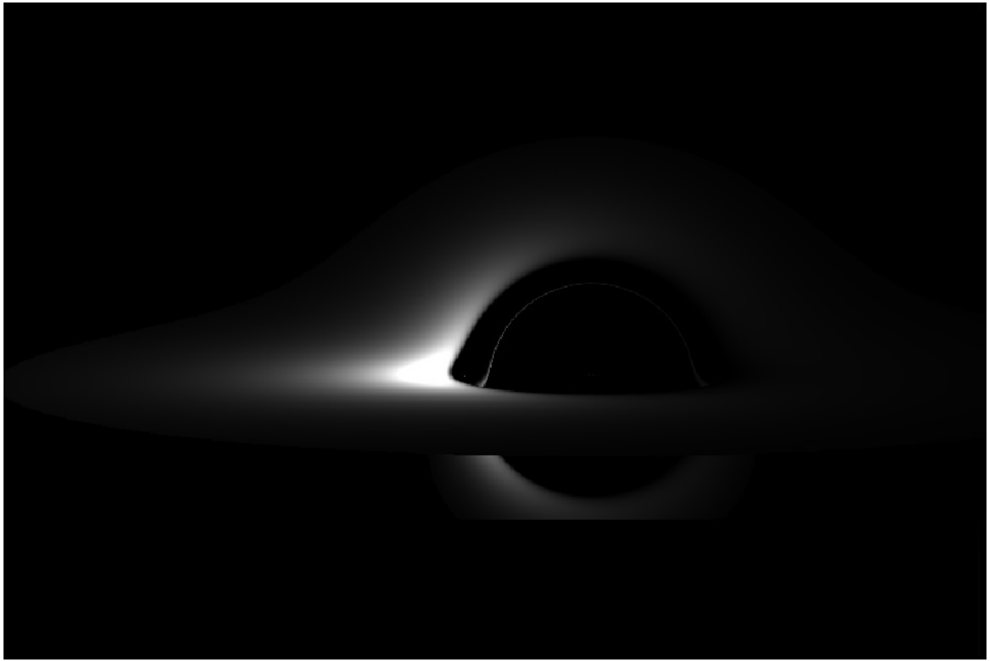


Imaging of a point source by a Schwarzschild black hole

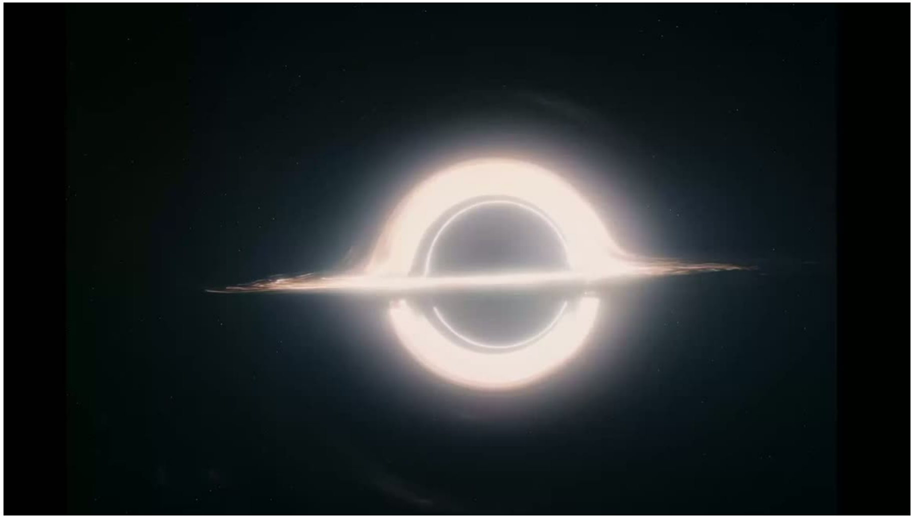




J.-P. Luminet (1979)



T. Müller (2012)



Interstellar (2014)

Shadow size of black-hole candidates

Object at the centre of our galaxy:

Mass $= 4.1 \times 10^6 \, M_\odot$

Distance $= 8.1 \, \mathrm{kpc}$

Angular diameter of the shadow by Synge's formula $\approx 54\mu$as

Corresponds to a grapefruit on the moon; shadow not yet observed

Object at the centre of M87:

Mass $= 6.5 \times 10^9 \, M_\odot$

Distance $= 16.4 \, \mathrm{Mpc}$

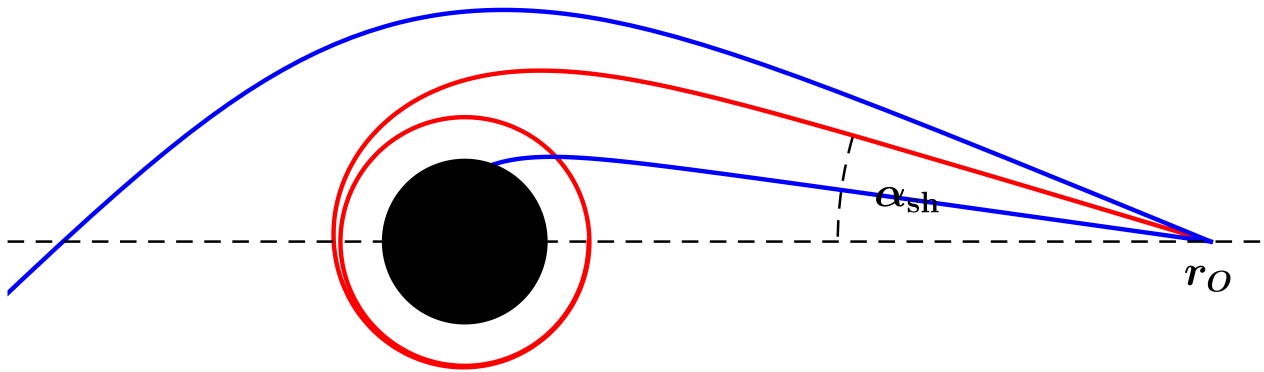
Angular diameter of the shadow by Synge's formula $\approx 42\mu$as

Observed by the Event Horizon Telescope Collaboration

Data taken in April 2017, released to the public in April 2019

Black hole impostor 1: Ultracompact star

Dark star with radius between $2m$ and $3m$



Shadow indistinguishable from Schwarzschild black hole

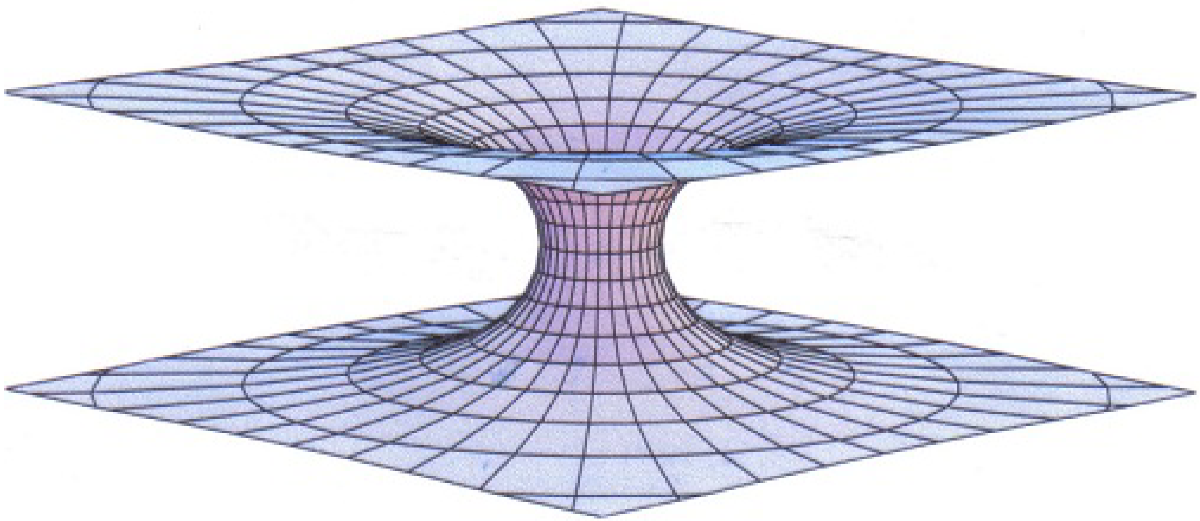
Ultracompact objects are unstable, see

V. Cardoso, L. Crispino, C. Macedo, H. Okawa, P. Pani: Phys. Rev. D 90, 044069 (2014)

Black hole impostor 2: Ellis wormhole

H. Ellis: J. Math. Phys. 14, 104 (1973)

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -c^2dt^2 + dr^2 + \left(r^2 + a^2\right)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$



Angular radius α of shadow: $\sin^2\alpha = \dfrac{a^2}{r_O^2 + a^2}$

2. Lensing by a Kerr black hole

Kerr metric

$$g_{\mu\nu}dx^\mu dx^\nu = \varrho(r,\vartheta)^2\left(\frac{dr^2}{\Delta(r)} + d\vartheta^2\right) + \frac{\sin^2\vartheta}{\varrho(r,\vartheta)^2}\Big(a\,dt - (r^2+a^2)d\varphi\Big)^2$$

$$-\frac{\Delta(r)}{\varrho(r,\vartheta)^2}\Big(dt - a\sin^2\vartheta\,d\varphi\Big)^2$$

$$\varrho(r,\vartheta)^2 = r^2 + a^2\cos^2\vartheta,$$

$$\Delta(r) = r^2 - 2mr + a^2\,.$$

$m = \dfrac{GM}{c^2}$ where $M =$ mass

$a = \dfrac{J}{Mc}$ where $J =$ spin

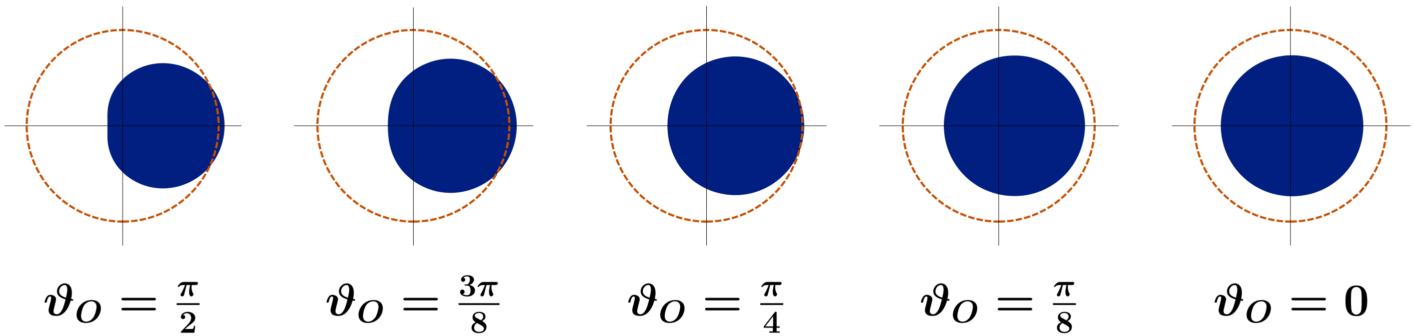
Shadow is no longer circular

Shape of the shadow of a Kerr black hole for observer at infinity:

J. Bardeen in C. DeWitt and B. DeWitt (eds.): "Black holes" Gordon & Breach (1973)

Shape and size of the shadow of Kerr black holes (and other black holes) for observer at coordinates (r_O, ϑ_O) (analytical formulas):

A. Grenzebach, VP, C. Lämmerzahl: Phys. Rev. D 89, 124004 (2014), Int. J. Mod. Phys. D 24, 1542024 (2015)



Shadow of Kerr black hole with $a = m$ for observer at $r_O = 5m$

Lightlike geodesics:

$$\varrho(r,\vartheta)^2\dot{t} = a\left(L - Ea\,\sin^2\vartheta\right) + \frac{(r^2+a^2)\left((r^2+a^2)E - aL\right)}{\Delta(r)},$$

$$\varrho(r,\vartheta)^2\dot{\varphi} = \frac{L - Ea\,\sin^2\vartheta}{\sin^2\vartheta} + \frac{(r^2+a^2)aE - a^2L}{\Delta(r)},$$

$$\varrho(r,\vartheta)^4\dot{\vartheta}^2 = K - \frac{(L - Ea\,\sin^2\vartheta)^2}{\sin^2\vartheta} =: \Theta(\vartheta),$$

$$\varrho(r,\vartheta)^4\dot{r}^2 = -K\Delta(r) + \left((r^2+a^2)E - aL\right)^2 =: R(r).$$

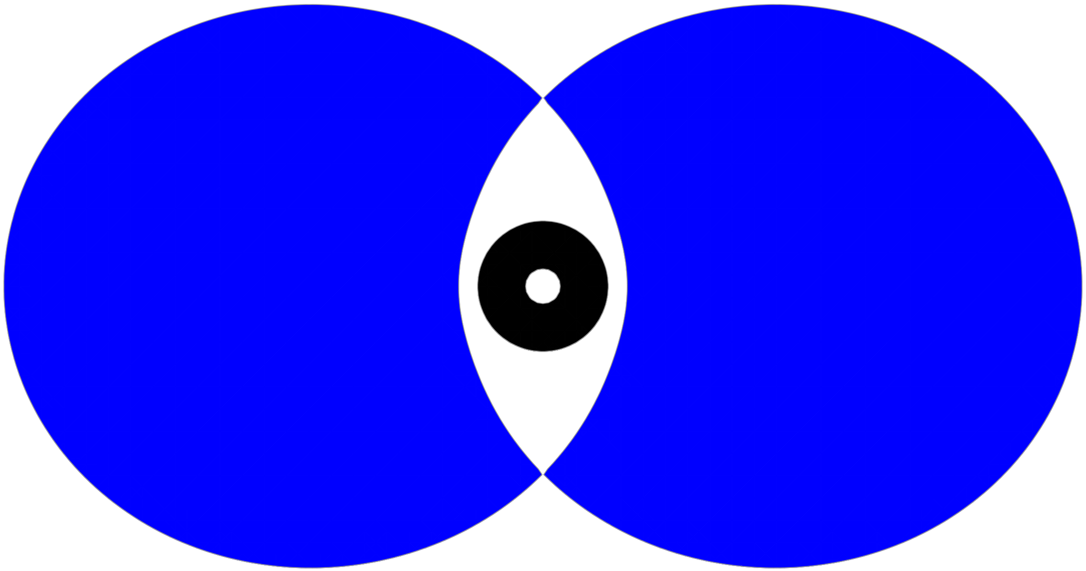
Spherical lightlike geodesics exist in the region where

$$R(r) = 0\,,\quad R'(r) = 0\,,\quad \Theta(\vartheta) \geq 0\,.$$

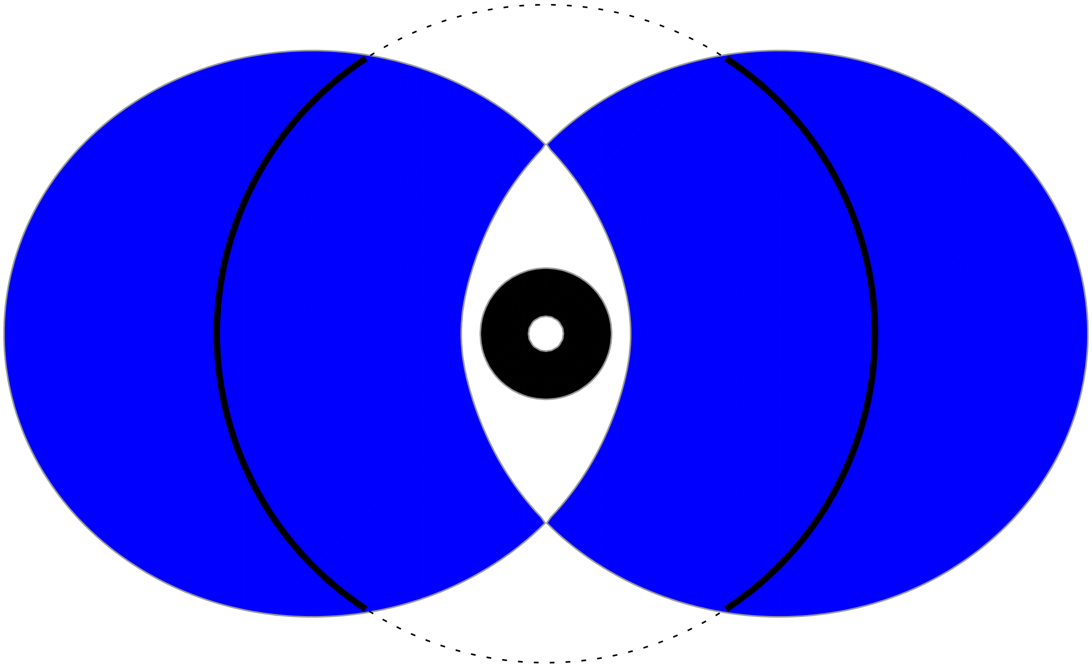
$$\left(2r\Delta(r) - (r-m)\,\varrho(r,\vartheta)^2\right)^2 \leq 4a^2r^2\Delta(r)\,\sin^2\vartheta$$

(unstable if $R''(r) \geq 0$)

Photon region for Kerr black hole with $a = 0.75\,m$



Photon region for Kerr black hole with $a = 0.75\, m$



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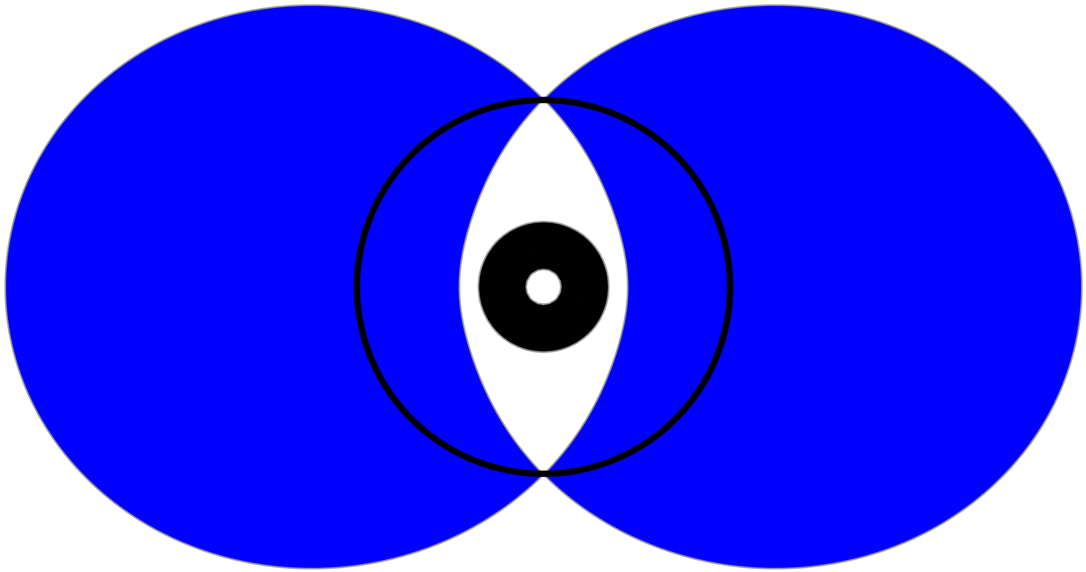

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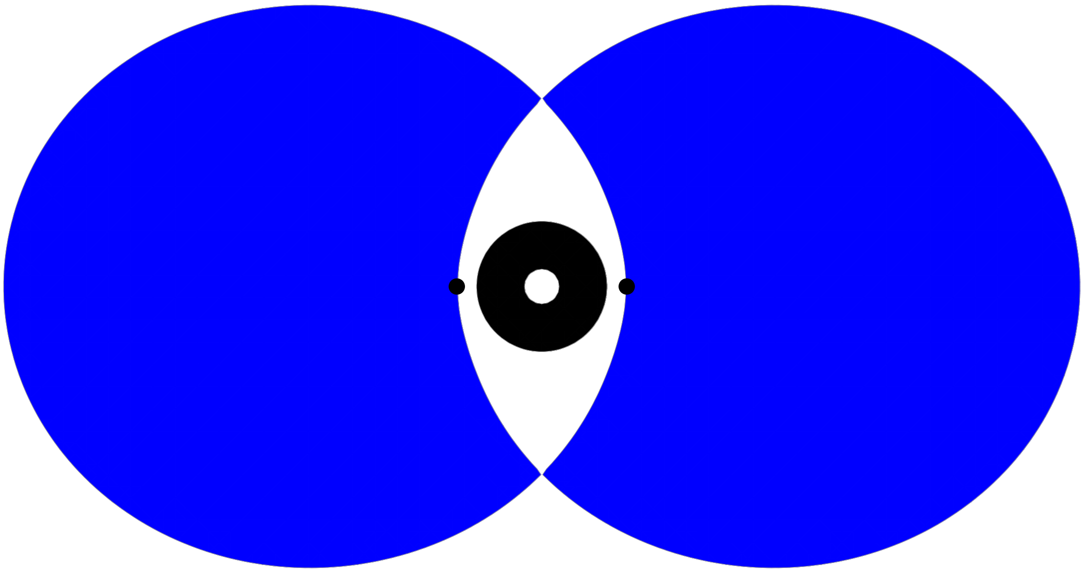

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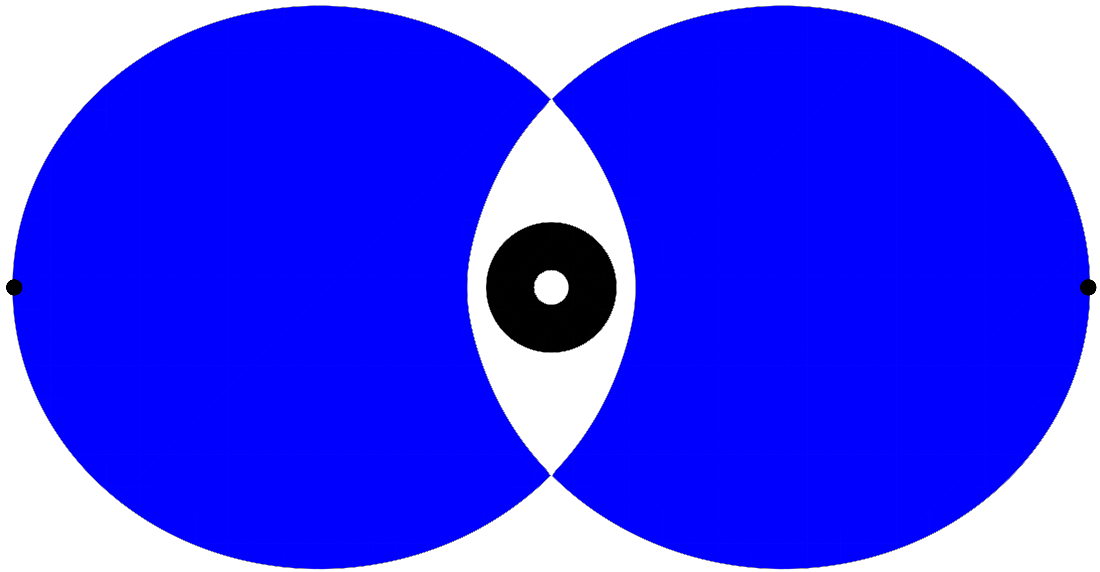


The shadow is determined by light rays that approach an unstable spherical lightlike geodesic.

Relation between constants of motion $\left(K_E = \dfrac{K}{E^2}, L_E = \dfrac{L}{E} - a\right)$ and celestial coordinates (θ, ψ):

$$\sin\theta = \frac{\sqrt{\Delta(r)\ K_E}}{r^2 - aL_E}\bigg|_{r=r_O}, \qquad \sin\psi = \frac{L_E + a\cos^2\vartheta + 2\ell\cos\vartheta}{\sqrt{K_E}\ \sin\vartheta}\bigg|_{\vartheta=\vartheta_O}$$

Constants of motion $\left(K_E, L_E\right)$ for limiting spherical lightlike geodesics:

$$K_E = \frac{16r^2\Delta(r)}{(\Delta'(r))^2}\bigg|_{r=r_p}, \qquad aL_E = \left(r^2 - \frac{4r\Delta(r)}{\Delta'(r)}\right)\bigg|_{r=r_p}$$

Gives boundary curve of the shadow $\theta(r_p), \psi(r_p)$ parametrised with r_p

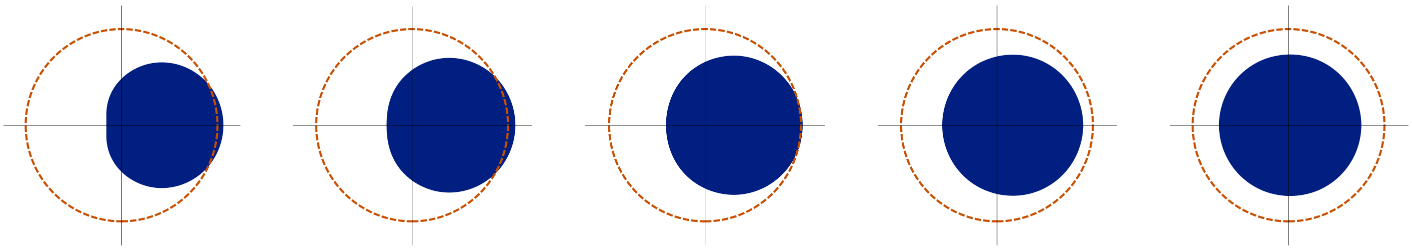
Analytic formula for shadow allows to extract parameters of the spacetime from the shape of the shadow

Vertical angular radius α_v of the shadow ($\vartheta = \pi/2$)

$$\sin^2 \alpha_v = \frac{27m^2 r_O^2 \big(a^2 + r_O(r_O - 2m) \big)}{r_O^6 + 6a^2 r_O^4 + 3a^2(4a^2 - 9m^2) r_O^2 + 8a^6} = \frac{27m^2}{r_O^2} \Big(1 + O(m/r_O) \Big)$$

A. Grenzebach, VP, C. Lämmerzahl: Int. J. Mod. Phys. D 24, 1542024 (2015)

Up to terms of order $O(m/r_O)$, Synge's formula is still correct for the vertical diameter of the shadow



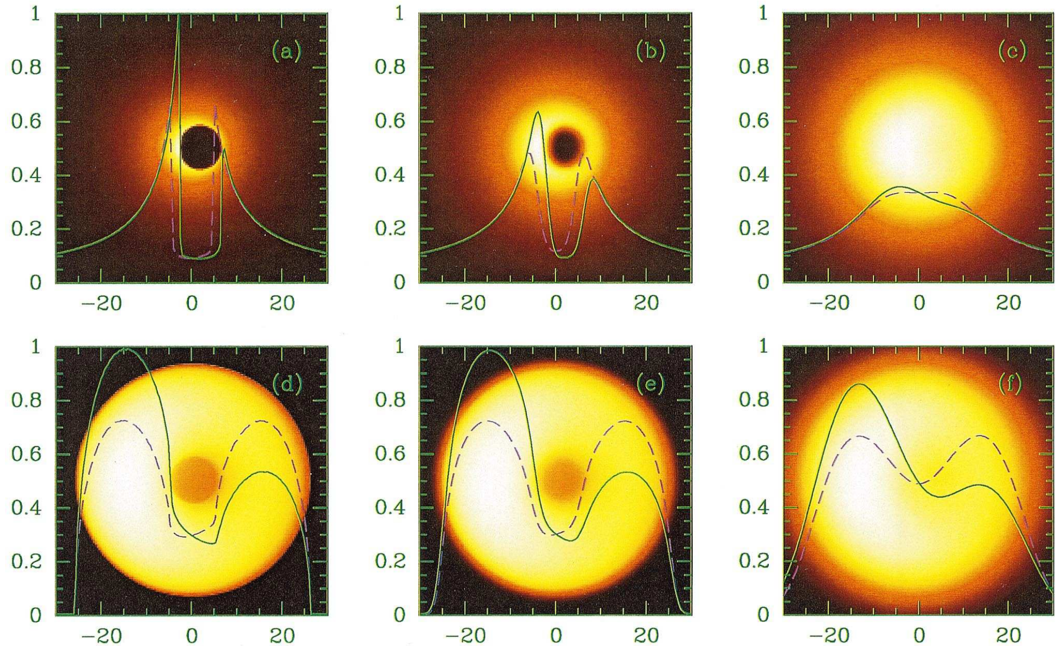
$\vartheta_O = \frac{\pi}{2}$ $\vartheta_O = \frac{3\pi}{8}$ $\vartheta_O = \frac{\pi}{4}$ $\vartheta_O = \frac{\pi}{8}$ $\vartheta_O = 0$

Shadow of Kerr black hole with $a = m$ for observer at $r_O = 5m$

3. Observability of the shadow

Kerr shadow with emission region and scattering
taken into account:

no scattering $\quad$ $\lambda = 0.6$ mm $\quad$ $\lambda = 1.3$ mm

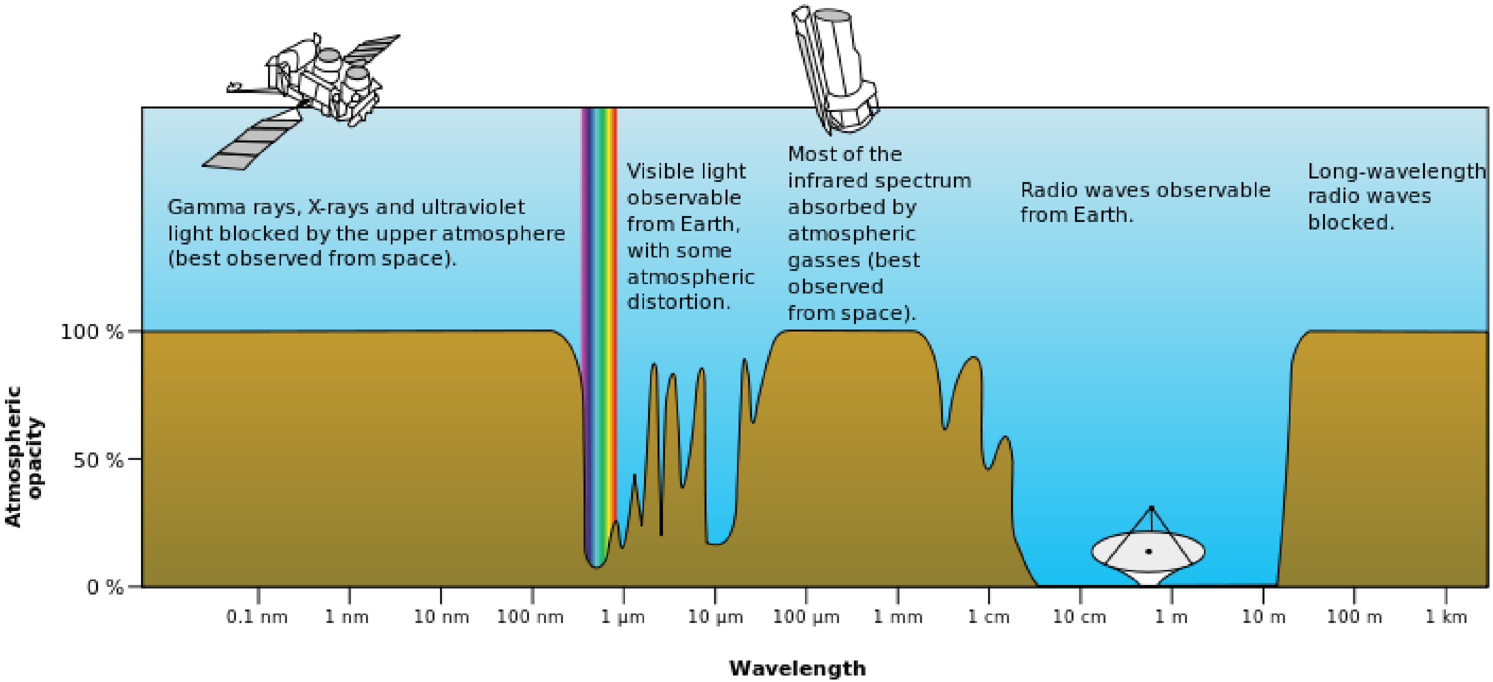


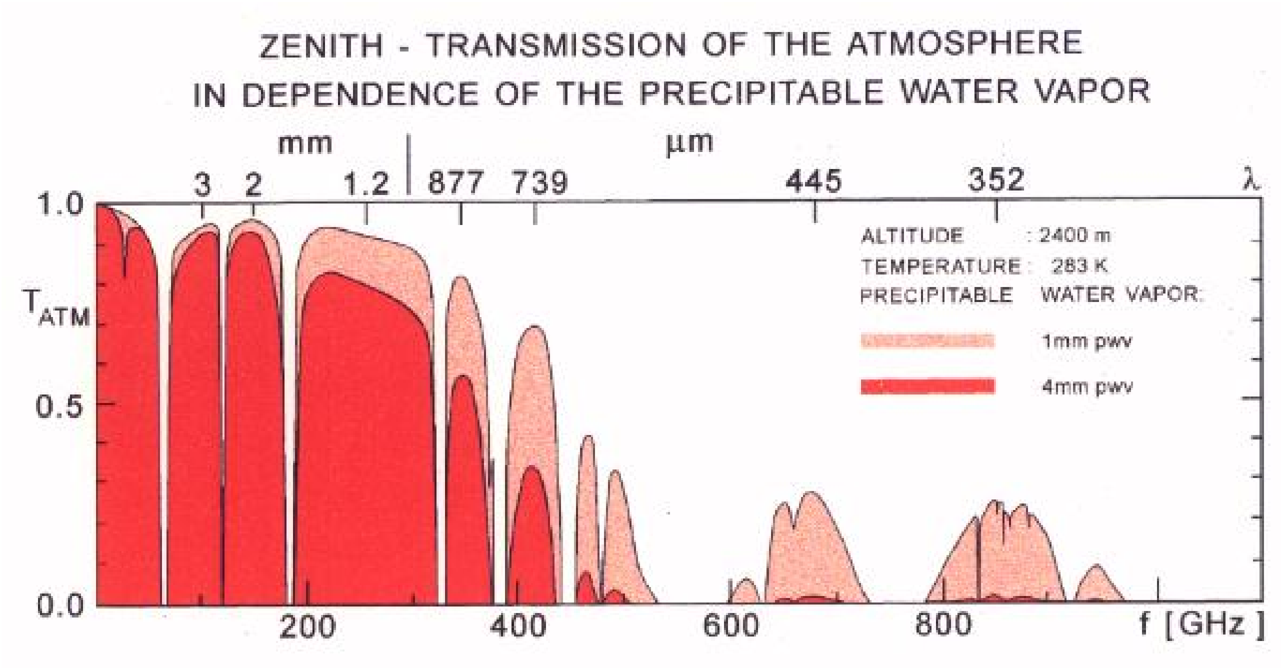
H. Falcke, F. Melia, E. Agol: Astrophys. J. 528, L13 (2000)

Observations should be done at sub-millimeter wavelengths

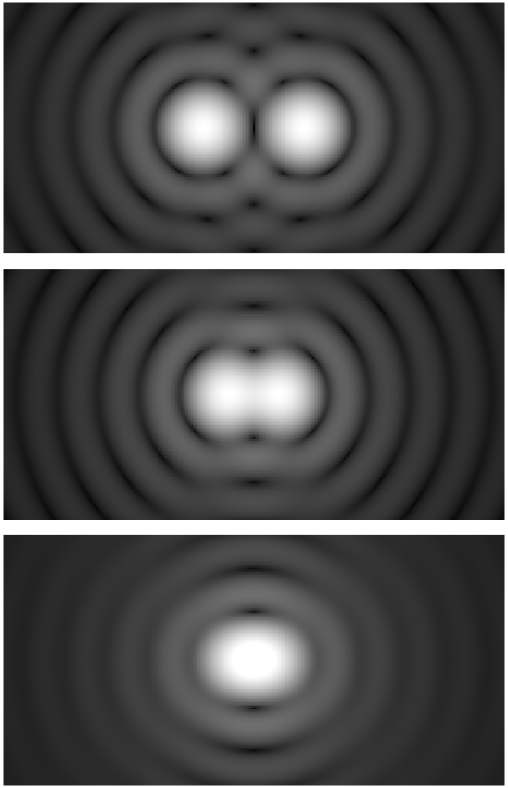
Opacity of the atmosphere

(at sea level under average atmospheric conditions)





Groundbased observations at $\approx 1\,\mathrm{mm}$ wavelength are possible with telescopes at high altitude (in dry areas)



Rayleigh criterion

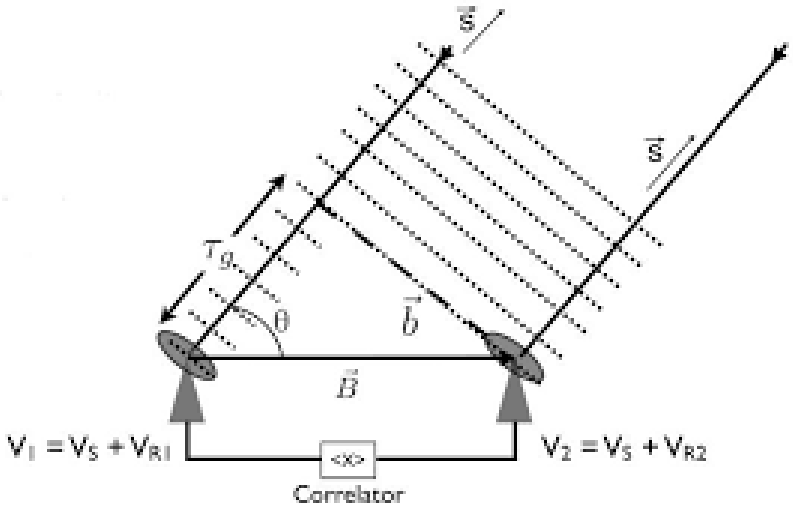
Angle θ that can be resolved by a telescope with circular aperture of diameter D at wavelength λ:

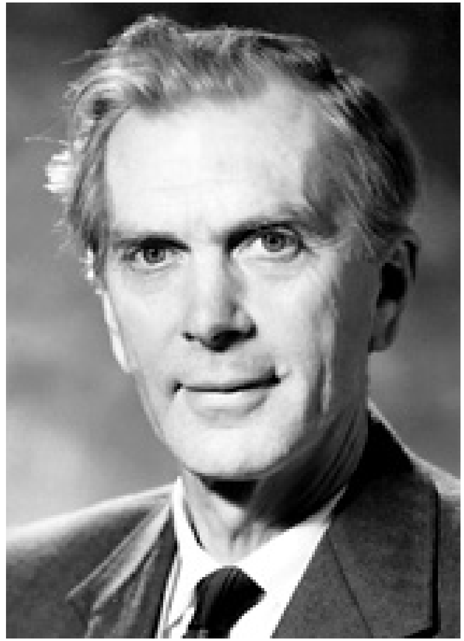
$$\theta = 1.22\,\frac{\lambda}{D}$$

$\lambda \approx 1\,$mm, $\theta \approx 20\mu$as

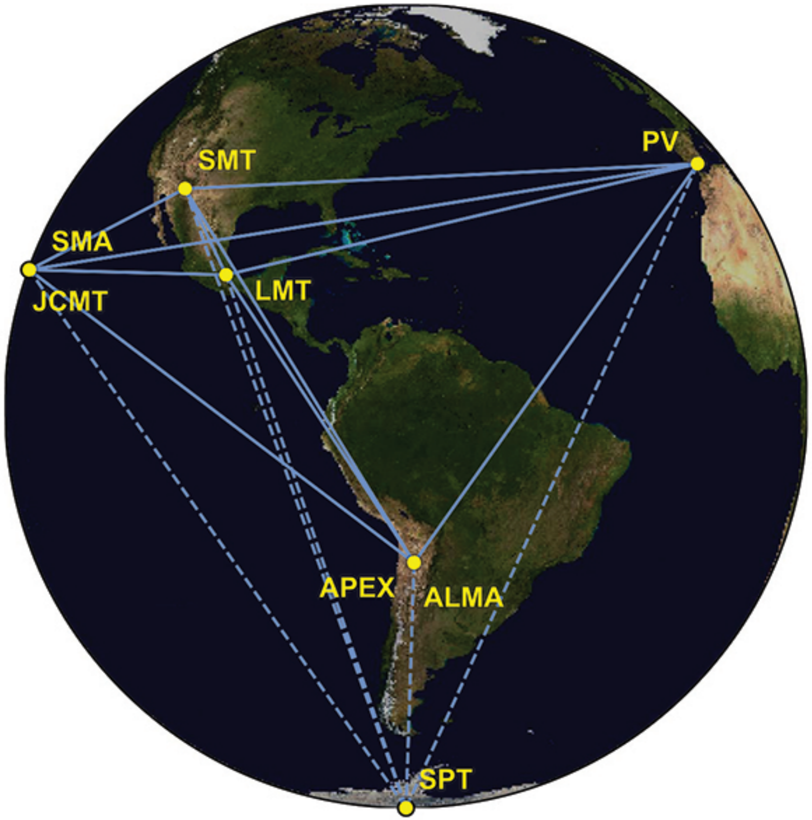
$D \approx$ diameter of Earth

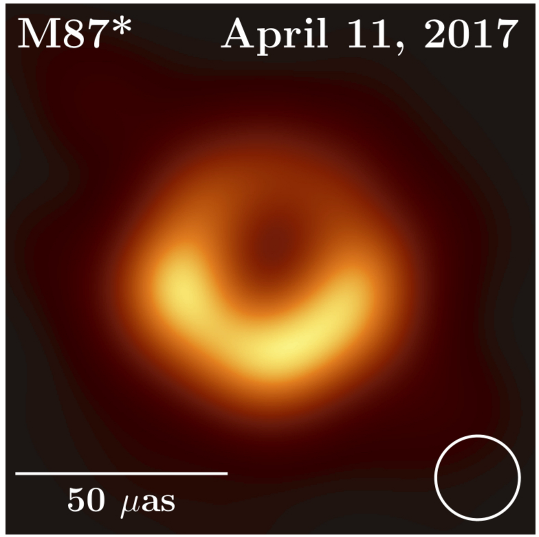
Aperture Synthesis



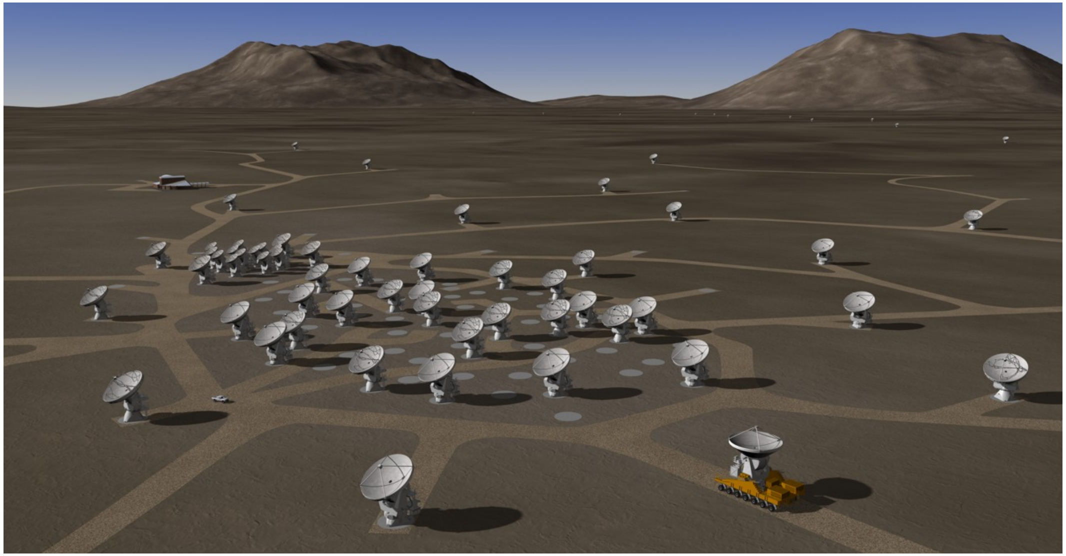


Martin Ryle
(1918-1984)



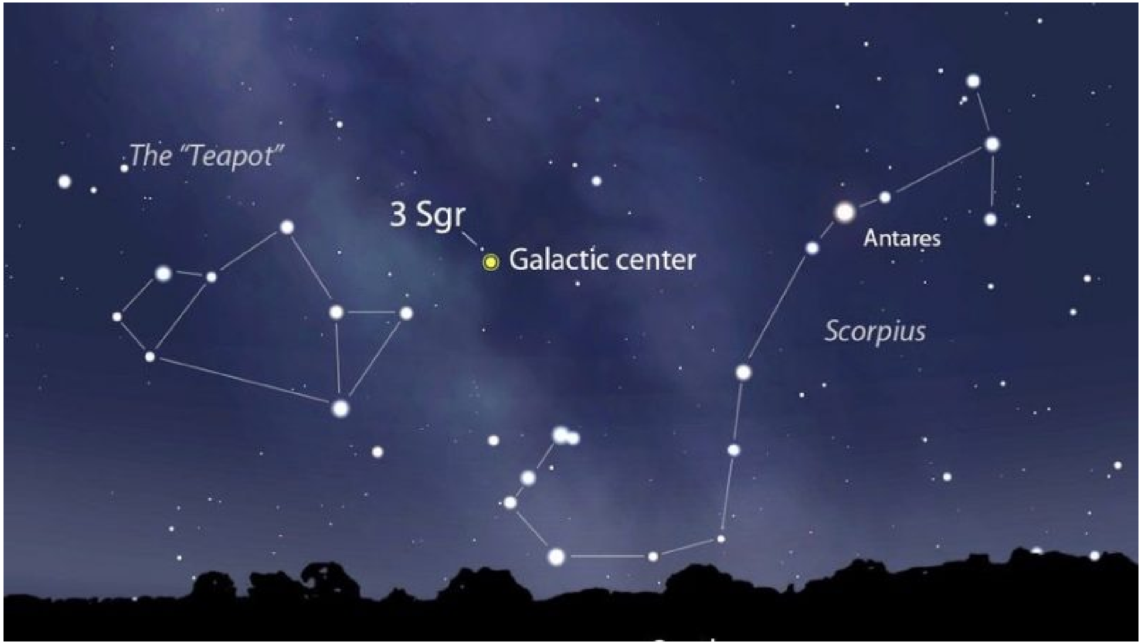


Event Horizon Telescope

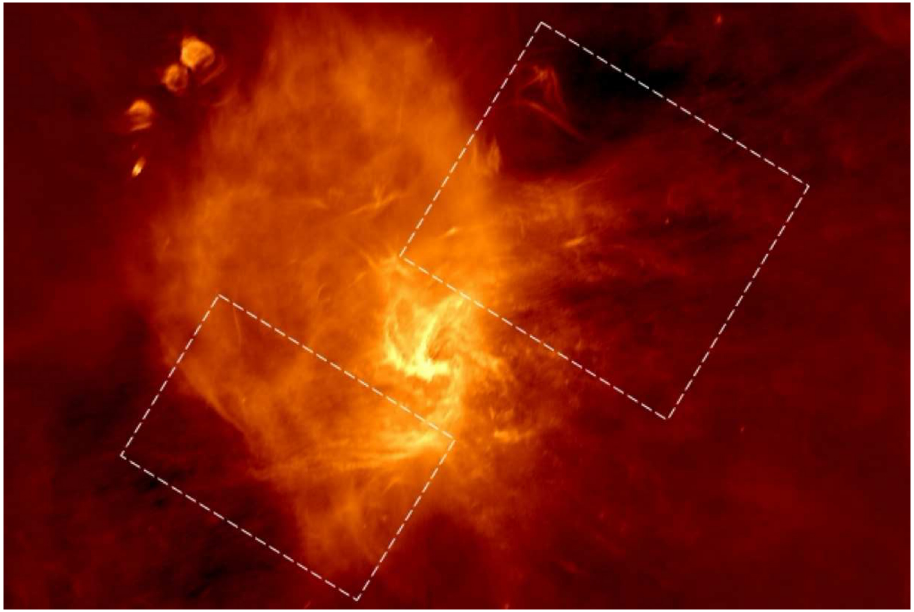


ALMA

Candidate 1: The centre of our Galaxy

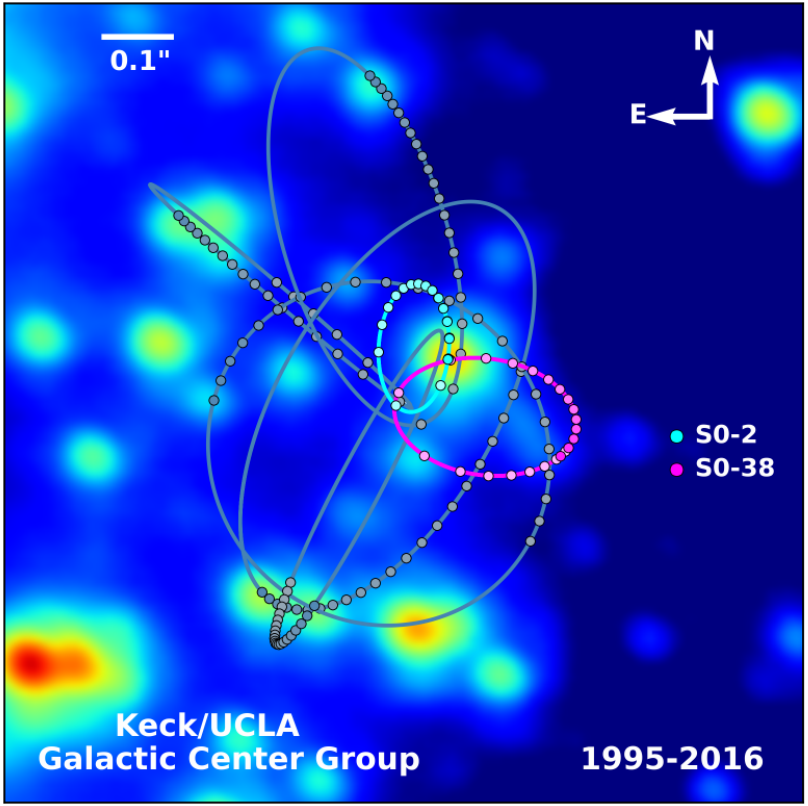


In the optical, the centre of our galaxy is hidden behind dust



In the radio one sees at the location of the centre of our galaxy a bright radio source, called Sgr A. The brightest spot, called Sgr A*, is located near the centre of the "minispiral" Sgr A West.
Picture taken with the Very Large Array at 5.5 GHz, diameter 13'

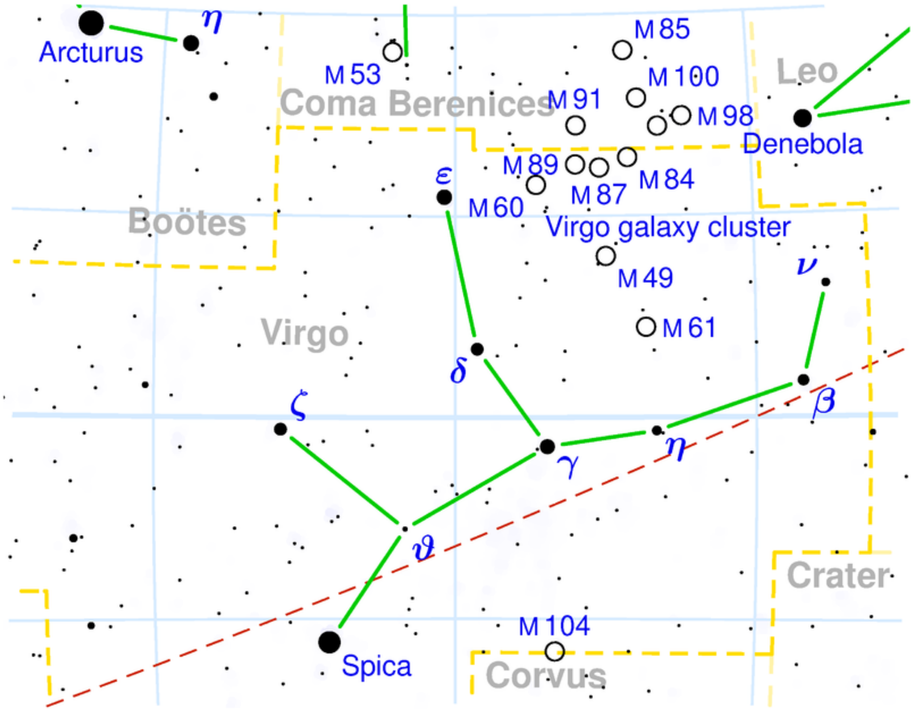
J.-H. Zhao, M. Morris and M. Goss, Astrophys. J. 817, 171 (2016)



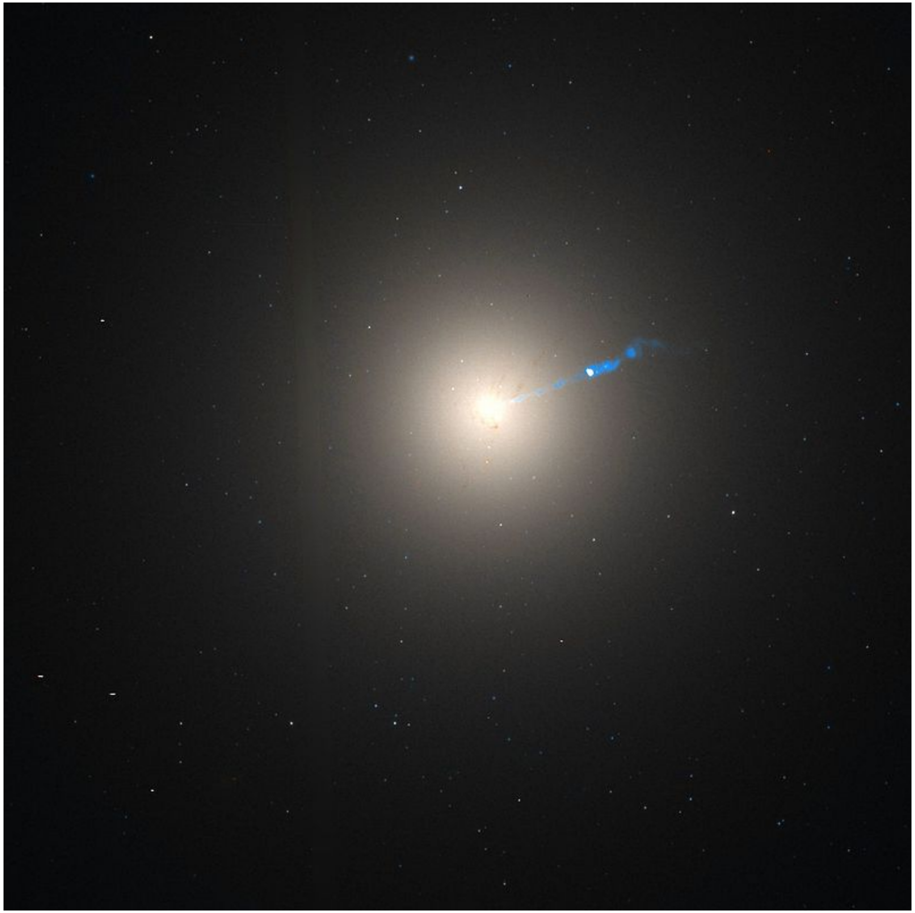
In the near infrared, one sees several stars (S0-2, SO-38 etc.) orbiting the centre of our galaxy. This allows to estimate the size and the mass of the central object

Picture: Andrea Ghez et al. (UCLA)

Candidate 2: The centre of M87

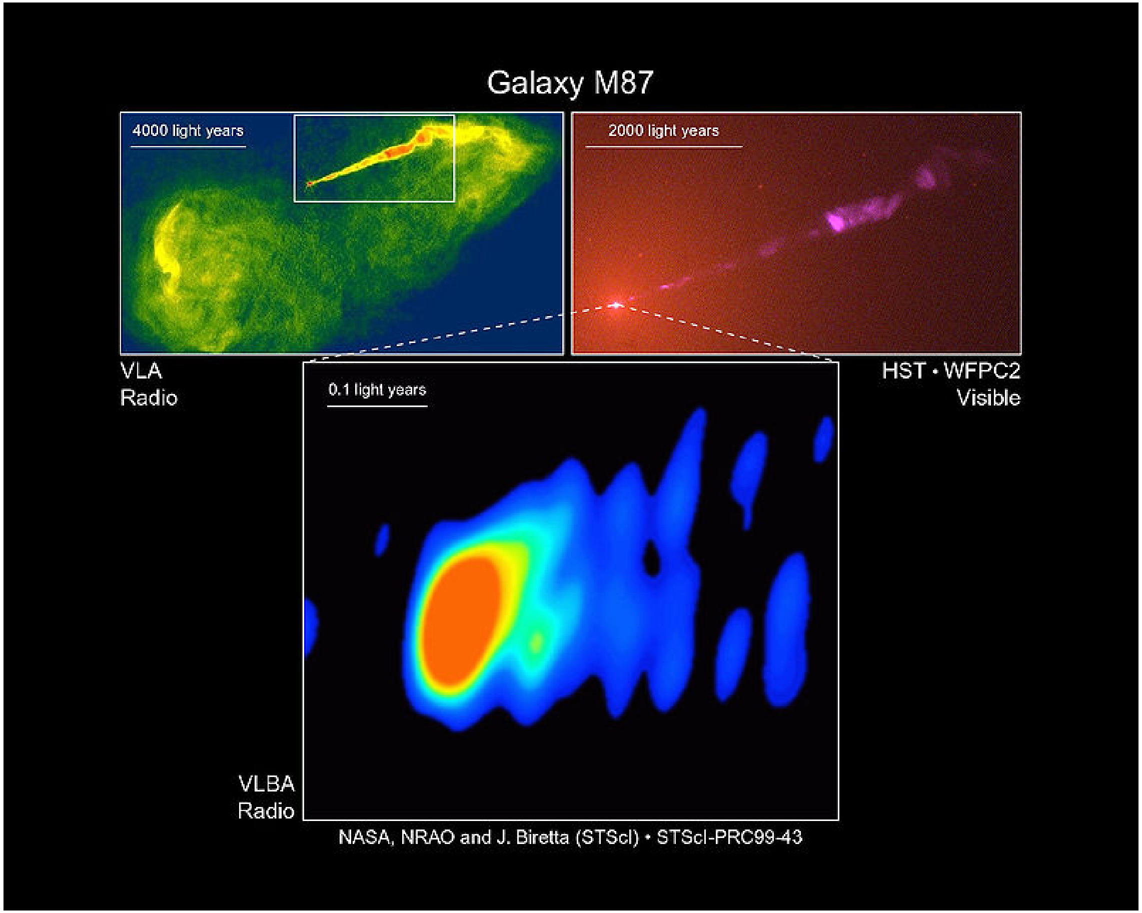


Location of M 87 in the sky

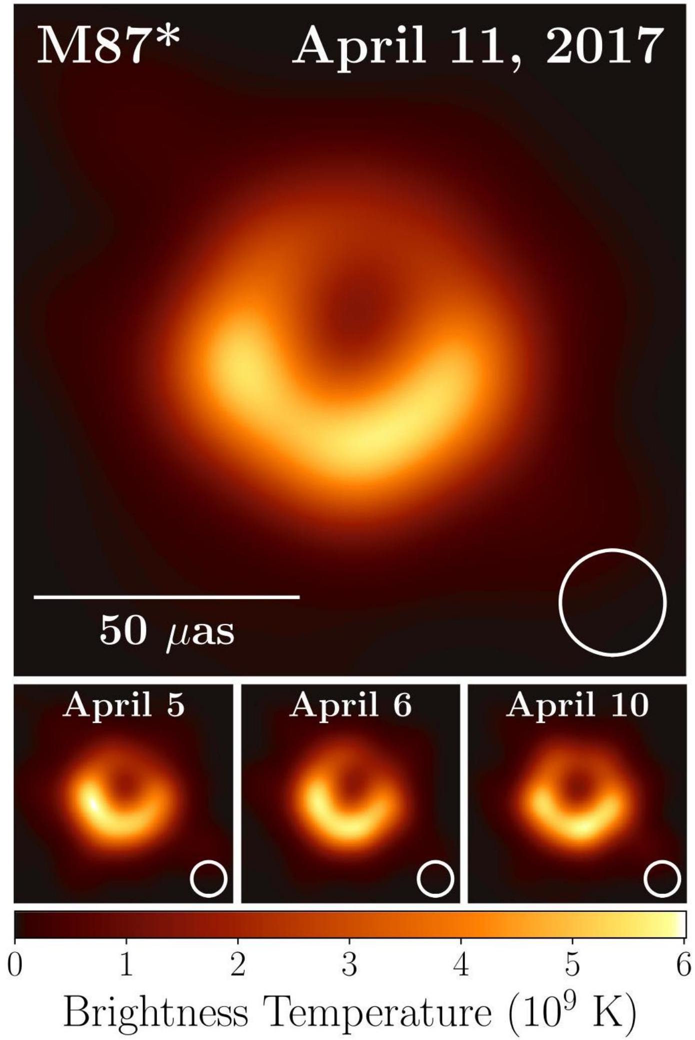


M 87 in the optical

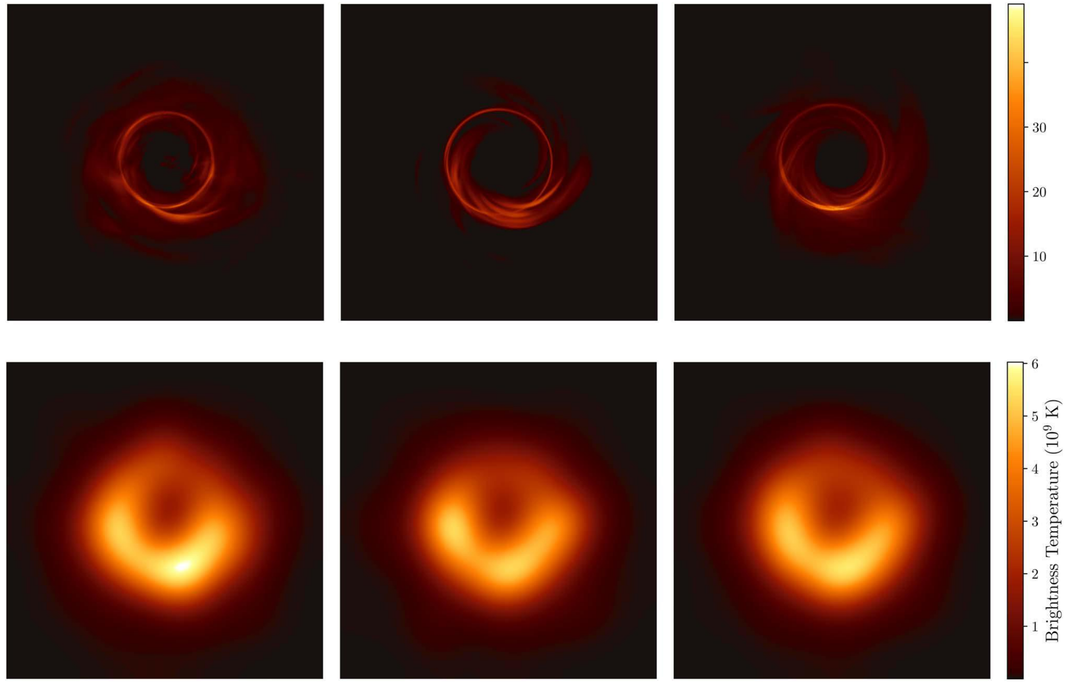
Picture: HST



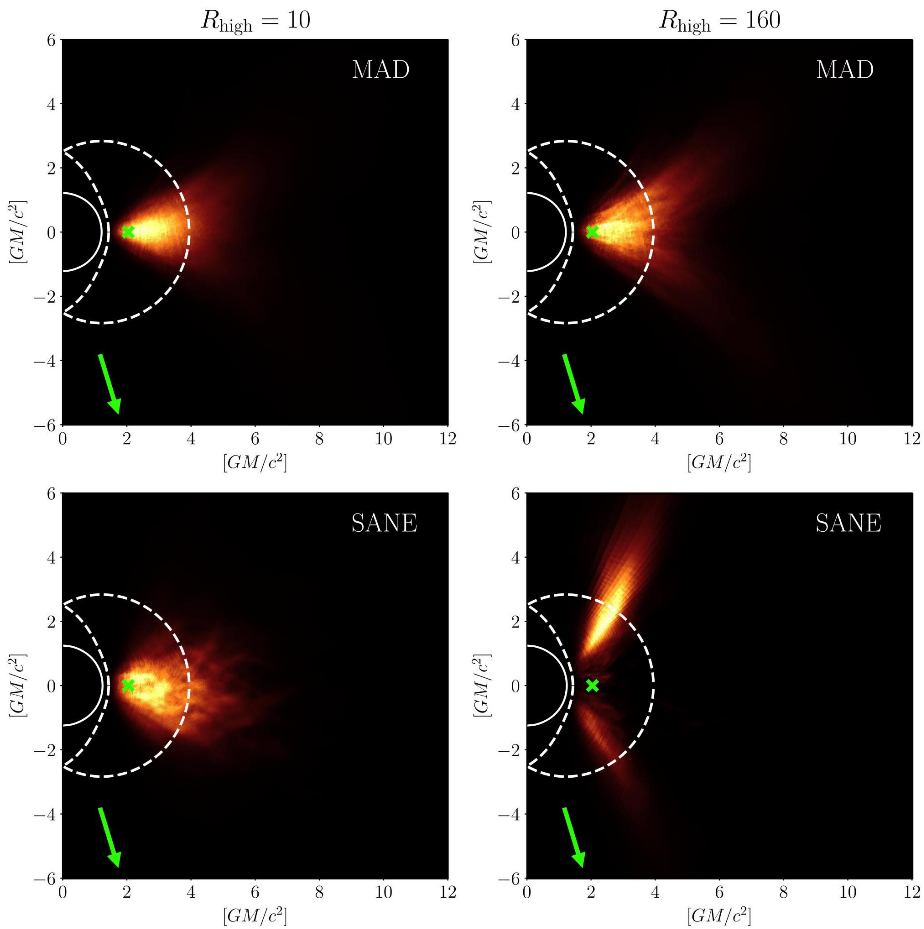
M 87 in the radio (Virgo A)



EHT Collaboration, Astrophys. J. Lett. 875, L1 (2019)



EHT Collaboration, Astrophys. J. Lett. 875, L1 (2019)



EHT Collaboration, Astrophys. J. Lett. 875, L5 (2019)