A notable quasi-relativistic wave equation and its relation to the Schrödinger, Klein-Gordon, and Dirac equations

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) \]

Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi_{\text{sch}}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_{\text{sch}}(x, t) \]

\[ \gamma_V = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \approx 1 \text{ when } V^2 \ll c^2 \]
References

Available at: http://luisgrave.com/id21.html


4. “Quasi-relativistic description of a quantum particle moving through one-dimensional piecewise constant potentials,” L. Grave de Peralta, Results in Physics, 18, 103318 (2020).


Acknowledgments


\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\hat{p}^2}{2m + \frac{\hat{p}^2}{2m + \ldots}}
\]
**Quasi-relativistic wave equation**

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) \]

**Schrödinger equation**

\[ i\hbar \frac{\partial}{\partial t} \psi_{Sch}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_{Sch}(x, t) \]

Quantum description of a free particle with mass \( m \) and spin-0

\[ \hat{E} = \hat{K} = i\hbar \frac{\partial}{\partial t}, \quad \hat{p} = -i\hbar \frac{\partial}{\partial x} \quad (1) \]

**Non-relativistic relation between \( K \) and \( p \):**

\[ K = \frac{p^2}{2m} \quad (2) \]

Substituting (1) into (2):

\[ i\hbar \frac{\partial}{\partial t} \psi_{Sch}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_{Sch}(x, t) \]

**Relativistic relation between \( K \) and \( p \):**

\[ K = \frac{p^2}{(\gamma_V + 1)m} \quad (3) \]

Substituting (1) into (3):

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) \]
Some results: 1D piecewise constant potentials

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V+1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) + U(x)\psi(x, t) \]  (1)

Looking for a solution of the form:

\[ \psi(x, t) = X_K(x)e^{\frac{iE't}{\hbar}}, \ E' = K + U \]  (2)

Then:

\[ \frac{d^2}{dx^2} X_K(x) + \kappa^2 X_K(x) = 0 \]  (3)

\[ \kappa = \frac{p}{\hbar} = \frac{1}{\hbar} \sqrt{(\gamma_V+1)mK} = \frac{1}{\hbar} \sqrt{(\gamma_V+1)m(E' - U)} \]  (4)

The values of \( \kappa \) are determined by the boundary conditions and \( K = E' - U \):

\[ K = \frac{\hbar^2 \kappa^2}{(\gamma_V+1)m} \]  (5)  \[ \text{But:} \quad K = (\gamma_V - 1)mc^2 \]  (6)

Then:

\[ \gamma_V^2 = 1 + \left( \frac{\hbar \kappa}{mc} \right)^2 \Rightarrow K = \frac{\hbar^2 \kappa^2}{1 + \sqrt{1 + \left( \frac{\hbar \kappa}{mc} \right)^2}}m \]  (7)

\[ \Delta K_{2,1}(L)/m_e c^2 \]

\[ \lambda_c = \frac{\hbar}{m_e c} \]

\[ L \text{ in } \lambda_c \text{ units} \]

\[ \Delta K_{2,1} \text{ calculated for } m = m_e \text{ using (continuous) quasi-relativistic wave and (dashed) Schrödinger equations.} \]

Infinite rectangular well

From Eq. (3): \( X_n(x) = \sqrt{\frac{2}{L}} \sin\left( \frac{n\pi}{L} x \right), \kappa_n = \frac{n\pi}{L}, n = 1, 2, \ldots \)

It is impossible to confine a single particle with mass in a point, this should be true for an electron, a quark, and probably may also be true for a black hole and the whole universe at the beginning of the Big Bang.
Some results: Hydrogen-like atoms

\[ i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -\frac{\hbar^2}{\mathcal{V}(r)+1} \nabla^2 \psi(\vec{r}, t) + U(r)\psi(\vec{r}, t) \quad (1) \]

In Eq. (1):

\[ U(r) = U_C(r) = -\frac{e^2}{2\pi\varepsilon_0 r} \quad (2) \]

Proceeding as it is done for solving the same problem when using the Schrödinger equation, it can be obtained the following results:

\[ E_{QR} = -\frac{\mu c^2}{\beta} [\beta - (2n + \Delta)\sqrt{\beta}] \quad (3) \]

In Eq. (3), \( n = 1, 2, \ldots \) is the principal quantum number and:

\[ \beta = 4n^2 + 4\alpha^2Z^2 + 4n\Delta + \Delta^2, \]

\[ \Delta = \left[ \left( 1 + \sqrt{(1 + 2l)^2 - 4\alpha^2Z^2} \right) - 2(l + 1) \right] \quad (4) \]

In Eq. (4), \( l = 0, 1, \ldots (n-1) \) is the orbital quantum number, and \( \alpha \) is the fine-structure constant. The following approximated formula is obtained by expressing Eq. (3) in powers of \( \alpha \) (up to \( \alpha^4 \)):

\[ E_{QR} \approx E_{Sch}Z^2 \left\{ 1 - \frac{\alpha^2}{n^2} \left[ \frac{3}{4} - \frac{n}{l+\frac{1}{2}} \right] \right\} \quad (5) \]

\[ E_{Sch} = -\left[ \frac{\mu}{2\hbar^2} \left( \frac{e^2}{4\pi\varepsilon_0} \right)^2 \right] \frac{Z^2}{n^2} = -\frac{\mu c^2}{4} \frac{\alpha^2Z^2}{n^2} \quad (6) \]

Comparison of the dependence on \( Z \) of the calculated energies for (a) \( n = 1 \) and \( l = 0 \), (b) \( n = 2 \) and \( l = 1 \). \( E \) was evaluated using (black continuous) Eq. (3), and the exact Dirac energies (blue dashed) with \( j = l + \frac{1}{2} \), and (red dot-dashed) with \( j = l - \frac{1}{2} \).
Some results: Hydrogen-like atoms

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -\frac{\hbar^2}{\left[\gamma V(r) + 1\right] m} \nabla^2 \psi(\vec{r}, t) + U(r) \psi(\vec{r}, t) \quad (1)$$

In Eq. (1):

$$U(r) = U_C(r) = -\frac{e^2 Z}{4\pi\varepsilon_0 r} \quad (2)$$

Proceeding as it is done for solving the same problem when using the Schrödinger equation, it was obtained for the ground state of Hydrogen-like atoms:

$$\mathcal{J}_{1,0}(r) \propto \left(\frac{c_1 \mu c r \hbar}{r}\right)^{B_{1,0}} e^{-A_{1,0} \frac{\mu c r \hbar}{r}} \quad (3)$$

In Eq. (3), $\mu$ is the electron reduced mass, and:

$$A_{1,0} = \frac{\sqrt{2 - 2\Delta}}{2}, \quad B_{1,0} = \frac{1 + \Delta}{2}, \quad C_{1,0} = \sqrt{2 - \sqrt{2 + 2\Delta}}$$

When $\alpha^2 Z^2 \ll 1$: $A_{1,0} \sim \alpha Z$, $B_{1,0} \sim 1$, $C_{1,0} \sim \alpha Z$, thus:

$$\mathcal{J}_{1,0}(r) \approx \mathcal{J}_{1,0,Sch}(r) \propto \frac{Z}{r_B} e^{\frac{Zr}{r_B}} \quad (4)$$

In Eq. (4), $r_B = \hbar/(\alpha \mu c)$ is the Bohr radius. Therefore, for the Hydrogen atom ($Z = 1$) the solution of Eq. (1) coincides with the solution of the Schrödinger equation. But for $Z = 100$, the Schrödinger equation strongly underestimate the confinement of the electron in the ground state.
Klein-Gordon equation

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi_{KG}(x, t) = \frac{\partial^2}{\partial x^2} \psi_{KG}(x, t) - \frac{m^2 c^2}{\hbar^2} \psi_{KG}(x, t)
\]

\[\hat{E} = i \hbar \frac{\partial}{\partial t}, \quad \hat{p} = -i \hbar \frac{\partial}{\partial x}\]  

(1)

For \( E > 0 \), Eq. (2) is equivalent to:

\[(E + mc^2)(E - mc^2) = p^2 c^2 \Rightarrow (E - mc^2) = \frac{p^2}{(\gamma V + 1)m} \]  

(3)

\[\text{Substituting (1) into (3):}\]

\[i \hbar \frac{\partial}{\partial t} \psi_{KG+}(x, t) = -\frac{\hbar^2}{(\gamma V + 1)m} \frac{\partial^2}{\partial x^2} \psi_{KG+}(x, t) + mc^2 \psi_{KG+}(x, t)\]  

(4)

Looking for a solution of Eq. (4) in the following form:

\[\psi_{KG+}(x, t) = \psi(x, t)e^{-i mc^2} \Rightarrow i \hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma V + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t)\]  

(5)

Quantum description of a free particle with mass \( m \) and spin-0 moving at relativistic speeds.
Dirac equation

\[ i\hbar \frac{\partial}{\partial t} \psi_D(\vec{r}, t) = c \left[ (\vec{\alpha} \cdot \vec{\beta}) \right] \psi_D(\vec{r}, t) + mc^2 \beta \] (1)

Each of the three components of the vector \((\vec{\alpha})\) and \((\vec{\beta})\) are \(4 \times 4\) Dirac's matrices. Each of the three components of the vector \((\vec{\beta})\) is the differential operator:

\[ \hat{p}_i = -i\hbar \frac{\partial}{\partial t}, \quad i = x, y, z \]

The bispinor \(\psi_D\) has four components; therefore, it can be represented using two spinors in the following way:

\[ \psi_D(\vec{r}, t) = \begin{pmatrix} \varphi(\vec{r}, t) \\ \chi(\vec{r}, t) \end{pmatrix} \] (2)

Proposing a solution of Eq. (1) of the following form:

\[ \psi_D(\vec{r}, t) = \begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} e^{-\frac{i}{\hbar}Et} \] (3)

Produces the following system of two time-independent spinor equations [Eqs. (4) and (5)]:

\[ c \left[ (\vec{\alpha} \cdot \vec{\beta}) \right] \chi = (E - mc^2)\varphi = K\varphi \] (4)

\[ c \left[ (\vec{\alpha} \cdot \vec{\beta}) \right] \varphi = (E + mc^2)\chi \] (5)

Each of the three components of the vector \((\vec{\alpha})\) is a \(2 \times 2\) Pauli's matrix. \(E + mc^2 > 0\) when \(E > 0\), thus when \(E > 0\), the bottom equation of Eq. (5) can be rewritten in the following way:

\[ \chi = \frac{c}{(E + mc^2)} \left[ (\vec{\alpha} \cdot \vec{\beta}) \right] \varphi = \frac{c}{(\gamma v + 1)mc} \varphi \] (6)

The two-component (spinor) time-independent quasi-relativistic wave equation for a free-particle particle with mass \(m\) and spin-1/2 can be obtained substituting Eq. (6) in Eq. (4):

\[ \frac{\left[ (\vec{\alpha} \cdot \vec{\beta}) \right]^2}{(\gamma v + 1)m} \varphi = -\frac{\hbar^2}{(\gamma v + 1)m} \nabla^2 \varphi = K\varphi \] (7)

Therefore, when \(E > 0\), each one of the two components of \(\varphi\) exactly satisfies the same time-independent quasi-relativistic wave equation, which corresponds to a free spin-0 particle with kinetic energy \(K\). Consequently, when \(V^2 << c^2\), each one of the two components of \(\varphi\) exactly satisfies the same time-independent Schrödinger equation:

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi_{Sch} = K\psi_{Sch} \] (8)
The two-component (spinor), time-independent, Pauli-like quasi-relativistic wave equation

(For a free electron moving through a constant magnetic field with magnitude \( B_{\text{ext}} \) pointing in the z direction)

\[
-\frac{\hbar^2}{(\gamma V+1)m}\nabla^2 \phi_P (\vec{r}) - \frac{2\mu_B}{(\gamma V+1)} B_{\text{ext}} \sigma_z \phi_P (\vec{r}) = E' \phi \tag{1}
\]

In Eq. (1) \( \mu_B = e\hbar/(2mc) \) is the Bohr magneton, and the spinor \( \phi_P \) is given by:

\[
\phi_P (\vec{r}) = \begin{pmatrix} \phi_{P+} (\vec{r}) \\ \phi_{P-} (\vec{r}) \end{pmatrix}
\]

Eq. (1) reduces to the corresponding time-independent Pauli equation when \( V^2 << c^2 \):

\[
-\frac{\hbar^2}{2m} \nabla^2 \phi_P (\vec{r}) - \mu_B B_{\text{ext}} \sigma_z \phi_P (\vec{r}) = E' \phi \tag{2}
\]

Equation (1) is the quasi-relativistic version of Eq. (2). When the electron moves slowly, \( \gamma V \sim 1 \), thus Eq. (1) coincides the non-relativistic Pauli equation. Eq. (1) includes two corrections to Eq. (2). First, includes the correct relativistic relation between \( K \) and \( p \). Second, as shown in the above figure, the energy difference corresponding to the two components of \( \phi_P \) is not independent of \( K \), as suggested by Eq. (2), but decreases by a factor of twice \( 2/(\gamma V+1) \) at quasi-relativistic energies. This relevant result could be easily tested experimentally.
**Some results: Hydrogen-like atoms**

Energies of the electron bound states in Hydrogen-like atoms using a perturbative approach based on the Schrödinger equation:

\[
E' = E_{Sch} \left( 1 + \Delta E_{K,Sch} + \Delta E_{D,Sch} + \Delta E_{SO,Sch} \right) \tag{1}
\]

In Eq. (1), there are three relativistic corrections. The first one is the relativistic correction to \( K \), the second one is the Darwin correction, and the third is the spin-orbit correction. However, when using a perturbative approach based in the quasi-relativistic wave equation:

\[
E' = E_{QR} \left( 1 + \Delta E_{D,QR} + \Delta E_{SO,QR} \right) \tag{2}
\]

In Eq. (2), there only are two relativistic corrections. The Darwin and spin-orbit corrections are given now by the following equations:

\[
\Delta E_{D,QR} = -k_D E_{QR} \frac{a^2 Z^2}{n}, \quad k_D = (\gamma_V + 1) \frac{n}{n+1} \tag{3}
\]

\[
\Delta E_{D,Sch} = -k_{SO} E_{QR} \frac{a^2 Z^2 j (j+1) - l (l+1) + \frac{3}{4}}{2 n \left( \frac{l+1}{2} \right) (l+1)}, \quad k_{SO} = \left( \frac{\gamma_V + 1}{2} \right)^{-(n-l+1)^2} \tag{4}
\]

In Eq. (4), \( j = l + \frac{1}{2} \) and \( j = l - \frac{1}{2} \). Therefore; the energies corresponding to the spectral lines are \( \Delta E_L = E'_{n, l, j} - E'_{n', l', j'} \)

Dependence on \( Z \) of \( \Delta E_L \) (in meV) calculated using (red, continuous) the exact values of the Dirac’s energies, (black, dot-dashed) Eq. (1), and (blue, dashed) Eq. (2) for (a) \( \alpha \)-Lyman doublet, (b) \( \alpha \)-Balmer doublet, (c) another example corresponding to the energy difference between two others emission lines involving a state with \( l \neq 0 \).
**About the relativistic invariance of luminal standing waves**

A luminal standing wave is formed by the superposition of two coherent beams of light traveling in the vacuum in opposite directions. The optical disturbance (ψ') of a luminal standing wave is described by the following equation:

$$\Psi'(x', t') = \sin(kx') \cos(w t')$$, \(k = 2\pi / \lambda\), \(w = 2\pi \nu\), \(\lambda \nu = c\). \hspace{1cm} (1)

However, due to the Doppler effect for light, one should expect that the optical disturbance (ψ) seen by an observer moving respect to the interference pattern should not be described by the above equation. This is because an observer moving parallel to the beams with speed \(V_o\) respect to the stationary interference fringes will see the superposition of two light beams with different frequencies. Therefore, the observer moving respect to the interference pattern would describe the optical disturbance resulting from the superposition of the light beams as the superposition of two plane waves with different frequencies traveling in opposite directions. It can be shown that the optical disturbance seen by the moving observer is described by the following expression:

$$\Psi(x, t) = \sin[k_{sb}(x - V_o t)] \cos[k_{sp}x - w_{sp}t]$$ \hspace{1cm} (2)

We found that the “cosine” factor in Eq. (2) have a superluminal phase velocity, which is not a “likable” feature. We also found that this “cosine” factor is a solution of the Klein-Gordon equation. I get curious about this, and I looked for a way to get rid of this unlikable feature. I found that the following wavefunction has a subluminal phase velocity:

$$\varphi(x, t) = \cos[k_{sp}x - w_{sp}t]e^{\frac{imc^2}{\hbar}}$$ \hspace{1cm} (3)

Moreover, I found that \(\varphi(x, t)\) satisfies the following equation:

$$i\hbar \frac{\partial}{\partial t} \varphi(x, t) = -\frac{\hbar^2}{(y \nu + 1)m} \frac{\partial^2}{\partial x^2} \varphi(x, t)$$

**Are there particles with mass and spin-0, which are formed from a superposition of two coherent beams of light?**

The rest is just curiosity + Chinese flu + good luck + more curiosity + Bill Poirier!
In units where $c = 1$: \[ K = \frac{p^2}{2m + K} \]

Poirier introduced the following recursive definition of the operator $K$:

\[
\tilde{K}_{\text{Poirier}} = \frac{\hat{p}^2}{2m + \tilde{K}} = \frac{\hat{p}^2}{2m + \frac{\hat{p}^2}{2m + \tilde{K}}} = \frac{\hat{p}^2}{2m + \frac{\hat{p}^2}{2m + \frac{\hat{p}^2}{2m + \tilde{K}}}} = \ldots
\]

\[
K = \frac{p^2}{(\gamma v + 1)m} = \frac{p^2c^2}{(\gamma v + 1)mc^2} = \frac{p^2c^2}{\gamma v mc^2 + mc^2} = \frac{p^2c^2}{2mc^2 + (\gamma v - 1)mc^2} = \frac{p^2c^2}{2mc^2 + K} \Rightarrow K(2mc^2 + K) = p^2c^2 \Rightarrow K^2 + 2mc^2K - p^2c^2 = 0
\]

Consequently:

\[
K = \frac{-2mc^2 + \sqrt{4m^2c^4 + 4p^2c^2}}{2} = \frac{\sqrt{m^2c^4 + p^2c^2} - mc^2}{2} = E - mc^2
\]